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ON CARDINAL SPLINE SMOOTHING.(U)

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ON CARDINAL SPLINE SMOOTHING

I. J. Schoenberg

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

The paper has two parts. In part I it describes the old results on the problem of smoothing a given bi-infinite sequence of equidistant data ([4], [5], [1], [2]). It includes in §3 the main results on the cardinal spline interpolation of equidistant data of power growth. All this is meant as background for the new developments in Part II.

In Part II we present a method of smoothing a sequence of equidistant data $\{y_\nu\}$ $(-\infty < \nu < \infty)$. It is based on the ideas of E. T. Whittaker and may be described as follows. In terms of the central B-spline $M_{2m}(x)$, defined by (7) of Part I, §3, we define the cosine polynomial $\phi_{2m}(u)$ by (26) of Part I, §3. Given a positive smoothing parameter ε, we define the sequence of "weights" $\{\omega_\nu(\varepsilon)\}$ by the Fourier expansion

$$\frac{1}{\phi_{2m}(u) + \varepsilon(2 \sin \frac{u}{2})^{2m}} = \sum_{-\infty}^{\infty} \omega_\nu(\varepsilon) e^{i\nu u} .$$

Let $\sum_{-\infty}^{\infty} |y_\nu| < \infty$. It is then shown that the cardinal spline

$$S(x;\varepsilon) = \sum_{-\infty}^{\infty} c_j M_{2m}(x-j),$$

where

$$c_j = \sum_{\nu=-\infty}^{\infty} y_\nu \omega_{j-\nu}(\varepsilon),$$

is the solution of the following minimum problem: <u>Among all functions</u> $f(x)$ <u>such that</u>

$$f^{(m)}(\varepsilon) \in L_2(\mathbb{R})$$

<u>to find</u> $f(x)$ <u>such that the functional</u>

$$J(f) \equiv \varepsilon \int_{-\infty}^{\infty} (f^{(m)}(x))^2 dx + \sum_{-\infty}^{\infty} (f(\nu) - y_\nu)^2 \quad \underline{\text{is to be minimal.}}$$

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Work Unit No. 6 - Spline Functions and Approximation Theory

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SIGNIFICANCE AND EXPLANATION

Let $\{y_\nu\}$ $(-\infty < \nu < \infty)$ be a given bi-infinite sequence of data that are in need of smoothing, i.e. replacement by a supposedly smoother sequence $\tilde{y}_\nu$. Let m be a natural number and ε a positive smoothing parameter. It is here shown how to construct a so-called cardinal spline function $S(x;\varepsilon)$ having the following properties.

1. $S(x;\varepsilon)$ reduces to a polynomial of degree $\leqq 2m-1$ in each of the intervals $(i, i+1)$, for all integers i.
2. $S(x;\varepsilon)$ has $2m-2$ continuous derivations for all real x.
3. If there exists a polynomial $P(x)$ of degree $\leqq 2m-1$ so that $y_\nu = P(\nu)$ for all integers ν, then $S(\nu;\varepsilon) = y_\nu$ for all ν.
4. If $\varepsilon = 0$, then $S(\nu;0) = y_\nu$, for all ν, provided that the y_ν do not grow too fast with $|\nu|$, in fact $y_\nu = O(|\nu|^\gamma)$ for some $\gamma > 0$.
5. If $\varepsilon > 0$, then the smoothed sequence $\{\tilde{y}_\nu\}$ is given by

   $$\tilde{y} = S(\nu;\varepsilon) \text{ for all } \nu .$$

The spline $S(x;\nu)$ is obtained by an optimal compromise (depending on ε) between two simultaneously irreconcilable situations:

(i) That $S(x;\varepsilon)$ should reduce to a polynomial of degree $\leqq m-1$,

(ii) That $S(\nu;\varepsilon) = y_\nu$ should hold for all ν.

If we increase the smoothing parameter ε, then we place more emphasis on the condition (i), while allowing $\{S(\nu;\varepsilon)\}$ to differ more from $\{y_\nu\}$.

In a future publication the author hopes to discuss the application of cardinal spline interpolation to the problem of approximating the Hilbert transform of a function $f(x)$ defined by the sequence of equidistant ordinates $\{f(\nu h)\}$ $(-\infty < \nu < \infty, h > 0)$. If this sequence is subject to experimental errors, it would seem advisable to perform on the sequence $\{f(\nu h)\}$ a preliminary smoothing operation by the method of the present paper.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON CARDINAL SPLINE SMOOTHING

By I. J. Schoenberg

Introduction. Let me thank Professor Ilio Galligani for asking me to visit Rome in May of 1977, where I gave a short 10-hours course on Cardinal Spline Interpolation at his Istituto per le applicazioni del Calcolo "Mauro Picone". He also asked me to prepare the above lectures for publication by his Institute. However, I noticed in Rome the strong current interest in the problem of smoothing of data prescribed at irregularly placed points in the plane. For this reason, and with Professor Galligani's approval, I decided to change the subject of this publication.

In the present form they describe the methods, with some changes to be mentioned below, whereby I solved the numerical problem assigned to me at the Ballistics Research Laboratories in Aberdeen, Maryland, during the second World War (see [4]). The problem was to smooth very extended equidistant tables of drag functions (or drag coefficients) by approximating them by very smooth functions that were easily computable with their first and second derivatives.

The first question to be answered was this: When may a "moving average" operation legitimately be called a smoothing formula? An answer is given in Section 1 of Part I. That it is a reasonable one is shown in Section 2 of Part I. These ideas were later greatly generalized by Fritz John in his important work on parabolic differential equations (see [3]). The connection is briefly mentioned in Section 2 of Part I.

The second essential ingredient of our results of Part II is the process of cardinal spline interpolation (see [9]). The required results are described in Section 3 of Part I.

The changes from my war-time approach are developed completely in Part II. This is the new contribution of the paper, and it arises if we adapt the idea of E. T. Whittaker (see [10] and [7]) to the problem of smoothing a bi-infinite sequence of equidistant data. The result is the cardinal spline smoothing process.

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Part I. Preliminaries

1. What is a smoothing formula? The several smoothing (or graduation) methods discussed in [10] are all of the "moving average" type. By this we mean that we are given a sequence of real and symmetric weights

(1) $$(a_n),\ a_n = a_{-n}\ (-\infty < n < \infty),\ \sum_{-\infty}^{\infty} a_n = 1,$$

which we apply to the data (x_n) to produce their smoothed version (y_n), by the formula

(2) $$y_n = \sum_{\nu=-\infty}^{\infty} a_{n-\nu} x_\nu,\ (n = 0, \pm 1, \cdots).$$

In other words, we convolute the sequences (a_n) and (x_n), an operation symbolically described by

(3) $$(y_n) = (a_n) * (x_n) .$$

By deriving a formula (2) according to a definite idea, such as a least squares formula, or Whittaker's formulae (see [10, Chapter XI]), we may reasonably expect that it will transform a given sequence (x_n) into a smoother sequence (y_n). This point becomes doubtful, however, in case the formula (2) was otherwise obtained, e.g., by some approximating procedure that is not strictly interpolatory. An example of such a procedure will be our cardinal spline smoothing of Part II. A criterion for (2) to be regarded as a smoothing formula was proposed by the author in [4, pp. 50-54] and proceeds as follows.

To begin with, we assume the Laurent series

(4) $$F(z) = \sum_{-\infty}^{\infty} a_n z^n, \quad (r < |z| < r^{-1}),$$

to converge in some ring containing the unit circle $|z| = 1$. Setting $z = e^{iu}$,

we call the even function

(5) $$\phi(u) = F(e^{iu}) = a_0 + 2a_1 \cos u + 2a_2 \cos 2u + \cdots$$

the characteristic function of the formula (2). The regularity of $F(z)$ on $|z| = 1$, implies that $\phi(u)$ is regular in a certain strip $|\mathrm{Im}\ u| < \alpha$, whence the valid Taylor expansion

(6) $$\phi(u) = \sum_{0}^{\infty} b_n u^n \ (|u| < \alpha) .$$

This expansion allows us to express easily an important property of (2): Its reproductive power, or degree of exactness.

We say that (2) has the degree of exactness $2m-1$, where m is a natural number, provided that (2) reproduces exactly, i.e., $y_n = x_n$ for all n, a sequence (x_n), if this sequence represents the values $P(n)$ of the polynomial of degree $2m-1$, but does not have this property for any higher degree. That the degree of exactness is always odd, follows from the symmetry condition (1). In terms of (6) we have the following easily established proposition:

The formula (2) has the degree of exactness $2m-1$, if and only if the expansion (6) is of the form

(7) $$\phi(u) = 1 - \lambda u^{2m} + \cdots \qquad (\lambda \neq 0).$$

An example. Let (1) be the sequence

$$a_0 = 5/8,\ a_1 = 1/4,\ a_2 = -1/16,\ a_n = 0 \ \text{ if } n > 2,$$

when the convolution (2) assumes the form

(8) $$y_n = (-x_{n-2} + 4x_{n-1} + 10x_n + 4x_{n+1} - x_{n+2})/16 .$$

From (5) we obtain

(9) $$\phi(u) = (5 + 4\cos u - \cos 2u)/8 = 1 - \frac{1}{16} u^4 + \cdots$$

and (7) is seen to hold with $m = 2$, $\lambda = 1/16$. It follows that the formula (8) has the degree of exactness 3.

May (8) be regarded as a smoothing formula? The general criterion is described by the following.

Definition. We say that (2) is a smoothing formula, provided that its characteristic function $\phi(u)$ satisfies

$$-1 < \phi(u) < 1 \quad \text{if} \quad 0 < u < 2\pi . \tag{10}$$

This condition evidently implies that the coefficient λ, appearing in (7), is positive. The characteristic function (9) is easily seen to satisfy (10), and so (8) is a smoothing formula according to our definition of the term.

The criterion (10) raises the following kind of question: Do the numerous smoothing formulae as given in [10] satisfy our criterion (10)? See Greville's paper [1] where some of these questions are answered affirmatively.

In [4, pp. 50-54] good arguments in support of the criterion (10) are presented. Even more convincing reasons were given in [5, Part I]. In the next section we reproduce these arguments as presented in [6, pp. 200-204].

2. The behavior of the iterates of a moving average formula: The problem of De Forest. A conclusive argument in support of the necessity of our condition (1.10) is furnished by the solution of the following problem first stated and attacked by Erastus L. De Forest (1834-1888). If we subject the given sequence (x_ν) n times in succession to the same transformation (1.2), we obtain a linear transformation

$$y_\nu^{(n)} = \sum_{k=-\infty}^{\infty} a_{\nu-k}^{(n)} x_k , \tag{1}$$

which is the n-fold iterate of (1.2). What is the asymptotic behavior of the coefficients of (1) as $n \to \infty$? This question was answered by De Forest and by

C. B. Dantzig (for references see [5]) for the case when all coefficients of (1.2) are non-negative, hence necessarily that $m = 1$ in (1.7). A general solution is as follows.

Let (1.2) be such that (1.1), (1.10), and (1.7), are satisfied, hence that $\lambda > 0$. Let

$$G_m(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-v^{2m}} \cos vx \, dv , \tag{2}$$

which is the normal frequency function

$$G_1(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{4}} \tag{3}$$

if $m = 1$, otherwise $(m = 2, 3, \cdots)$ $G_m(x)$ is an entire function having infinitely many zeros, all real.

The coefficients of (1) satisfy the asymptotic relations

$$a_\nu^{(n)} = (\lambda n)^{-\frac{1}{2m}} G_m\left(\nu(\lambda n)^{-\frac{1}{2m}}\right) + o\left(n^{-\frac{1}{2m}}\right) \quad \text{as } n \to \infty , \tag{4}$$

where the "little o" symbol holds uniformly for all integers ν.

For a proof see [5, Part I], where it is also shown by examples that (4) no longer holds if the equality sign is allowed in (1.10), and that the coefficient $a_0^{(n)}$ diverges exponentially to $+\infty$, as $n = 2k$ tends to infinity through even values, if the inequalities (1.10) are reversed anywhere in the interval $0 < u < 2\pi$.

The following discussion, while not directly related to our subject of smoothing, will show the connection of the asymptotic relation (4) with the wider field of parabolic differential equations.

Observe that (2) implies that

$$(5) \qquad U(x,t) = t^{-\frac{1}{2m}} G_m\left(x\, t^{-\frac{1}{2m}}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tv^{2m} + ixv}\, dv, \quad (t > 0).$$

The function under the integral sign is immediately seen to satisfy for all v, the differential equation

$$(6) \qquad \frac{\partial u}{\partial t} = (-1)^{m+1} \frac{\partial^{2m} u}{\partial x^{2m}} \quad (x \in \mathbb{R}, \quad t > 0),$$

which reduces to the familiar heat equation if $m = 1$. It follows that also $U(x,t)$, defined by (5), is a solution of (6) in the upper half-plane $t > 0$. On the other hand, applying to (2) Fourier's inversion formula and setting $v = 0$, we find that

$$\int_{-\infty}^{\infty} G_m(x)\, dx = 1.$$

These remarks imply the following: <u>If</u> $f(x)$ <u>is continuous and</u> $o(|x|^{-2})$, say, as $|x| \to \infty$, <u>then</u>

$$(8) \qquad u(x,t) = t^{-\frac{1}{2m}} \int_{-\infty}^{\infty} G_m\left\{(x - v)\, t^{-\frac{1}{2m}}\right\} f(v)\, dv, \quad (t > 0),$$

<u>is a solution of the differential equation (6) satisfying the boundary condition</u>

$$(9) \qquad \lim_{t \to 0+} u(x,t) = f(x).$$

This particular solution $u(x,t)$ may now also be approximated by the following numerical procedure: Draw in the (x,t)-plane the rectangular lattice of points

$$(\nu\Delta x, n\Delta t) \quad (\nu = 0, \pm 1, \ldots; n = 0, 1, 2, \ldots).$$

Define on it a lattice function $u_{\nu,n}$ by starting with

$$u_{\nu,0} = f(\nu\Delta x),$$

and computing the values along each horizontal line from those on the line below it, by means of the transformation (1.2). This evidently amounts to iterating (1.2), and after n steps we obtain

$$(10) \qquad u_{\nu,n} = \sum_{\mu=-\infty}^{\infty} a_{\nu-\mu}^{(n)} f(\mu\Delta x).$$

For any given x and $t > 0$, (10) will go over into (8) if we do the following: <u>We first connect the mesh-sizes</u> Δx <u>and</u> Δt <u>by the relation</u>

$$(11) \qquad \Delta t = \lambda(\Delta x)^{2m}.$$

<u>If the integers ν and n are such that</u>

$$\nu\Delta x \to x, \text{ and } n\Delta t \to t \quad \text{as } \Delta x \to 0.$$

<u>then</u>

$$u_{\nu,n} \to u(x,t).$$

This follows readily from (10) and (8), in view of the asymptotic relation (4): (10) differs from a Cauchy-Riemann sum for the integral (8), by a quantity that tends to zero due to the uniformity in ν of the error term of (4).

It is interesting to note that it does not matter which formula (1.2) we use in this construction, as long as it is of the degree of exactness $2m-1$, i.e., it satisfies (1.7), and above all that it satisfies the stability condition (1.10), the term "stability" meaning here stability on iteration. For the general theory of F. John, of which the equation (6) is a special example, see [3].

In this section we dealt exclusively with formulae (1.2) which satisfy the symmetry relation. In [2] T. N. E. Greville dealt with the more difficult case of unsymmetric formulae.

3. <u>Cardinal spline interpolation</u> (See [9, Lectures 1-4]). The problem of <u>cardinal interpolation</u> is to find solutions $f(x)$ of the interpolation problem

$$(1) \qquad f(\nu) = y_\nu, \quad \text{for all integers } \nu,$$

where (y_ν) are the data. A formal solution is furnished by the series

(2) $$f(x) = \sum_{-\infty}^{\infty} y_\nu \frac{\sin \pi(x-\nu)}{\pi(x-\nu)}$$

investigated in 1908 by de la Vallée Poussin, also later by E. T. Whittaker, who called it the cardinal series. The difficulty with (2) is the slow decay of the function $\frac{\sin \pi x}{(\pi x)}$ as $x \to \infty$. A much simpler solution of (1) is the piecewise linear interpolant $S_1(x)$ given by

(3) $$S_1(x) = \sum_{-\infty}^{\infty} y_\nu M_2(x-\nu),$$

where $M_2(x)$ is the roof function defined by

$M_2(x) = x + 1$ in $[-1,0]$, $M_2(x) = 1 - x$ in $[0,1]$ and $M_2(x) = 0$ if $|x| > 1$.

The purpose of cardinal spline interpolation is to bridge the gap between the piecewise linear $S_1(x)$ defined by (3), and the cardinal series (2). It aims at retaining some of the sturdiness and simplicity of (3), at the same time capturing some of the smoothness and sophistication of (2).

Let m be a natural number, and let

(4) $$\mathcal{S}_{2m-1} = \{S(x)\}$$

be the class of cardinal splines $S(x)$ of degree $2m-1$ defined by the two conditions:

(5) $$S(x) \in C^{2m-2}(\mathbb{R}),$$

(6) The restriction of $S(x)$ to every unit interval $(\nu, \nu+1)$, where ν is an integer, is a polynomial of degree $\leq 2m-1$.

For $m = 1$ we find $\mathcal{S}_1$ to be identical with the class (3) of continuous piecewise linear functions. Observe that $\mathcal{S}_{2m-1}$ contains the class of polynomials of degrees not exceeding $2m-1$.

The role of the roof-function $M_2(x)$, of (3), is taken over by the so-called central B-spline $M_{2m}(x)$: Writing $x_+ = \max(x,0)$, it may be defined by*

(7) $$M_{2m}(x) = \frac{1}{(2m-1)!}\,\delta^{2m} x_+^{2m-1} = \frac{1}{(2m-1)!} \sum_{\nu=0}^{2m} (-1)^\nu \binom{2m}{\nu} (x+m-\nu)_+^{2m-1}.$$

Clearly $M_{2m}(x) \in \mathcal{S}_{2m-1}$; we also find that $M_{2m}(x) > 0$ in its support $-m < x < m$. The B-spline should be familiar in view of the fundamental identity

$$\delta^{2m} f(0) = \int_{-m}^{m} M_{2m}(x)\, f^{(2m)}(x)\, dx,$$

which also shows that $\int M_{2m}(x)\,dx = 1$ if we choose $f(x) = x^{2m}$.

The representation (3) also generalizes, and every $S(x) \in \mathcal{S}_{2m-1}$ admits a unique representation

(8) $$S(x) = \sum_{-\infty}^{\infty} c_\nu M_{2m}(x-\nu),$$

where the c_ν are constants. This is the so-called standard representation. The converse is clear: Every series (8) furnishes an element of $\mathcal{S}_{2m-1}$. We now try to solve the interpolation problem (1) by elements of $\mathcal{S}_{2m-1}$. In this direction there are two different kinds of results.

A. The data (y_ν) are of power growth (See [8]). We say that the sequence (y_ν) is of power growth, and write

(9) $$(y_\nu) \in PG,$$

provided that

(10) $$y_\nu = O(|\nu|^\gamma) \text{ as } \nu \to \pm\infty, \quad \text{for some } \gamma \geq 0.$$

Similarly, we write

(11) $$f(x) \in PG,$$

---

* δ denotes the finite difference taken centrally.

provided that $f(x) = O(|x|^{\gamma_1})$ as $x \to \pm\infty$, for some $\gamma_1 \geq 0$.

Below we exclude the trivial case when $m = 1$, since our problem is solved by (3) without any restriction on the (y_ν).

Theorem 1. If the sequence (y_ν) is of power growth, then the interpolation problem

(12) $$S(\nu) = y_\nu \quad \text{for all } \nu,$$

has a unique solution $S(x)$ such that

(13) $$S(x) \in \mathcal{S}_{2m-1} \cap PG.$$

The assumption (9) of Theorem 1 is a rough one; it admits, e.g., all bounded sequences (y_ν), with $\gamma = 0$ in (10). The second assumption to which we now pass, is much more selective, and takes into account the finer structure of the sequence; in fact it admits only a narrow subclass of the sequences of PG. As usual, with stronger assumptions, stronger conclusions are possible: The interpolant $S(x)$ will exhibit an important extremum property.

B. The case when $\sum_{-\infty}^{\infty} |\Delta^m y_\nu|^2 < \infty$ (See [9, Lecture 6]). We introduce the classes of sequences and functions as follows:

(14) $$\ell_2^m = \{(y_\nu);\ \sum_{-\infty}^{\infty} |\Delta^m y_\nu|^2 < \infty\},$$

(15) $L_2^m = \{f(x);\ f, \ldots, f^{(m-1)}$ are absolutely continuous, $f^{(m)}(x) \in L_2(\mathbb{R})\}$.

Of course ℓ_2^0 and L_2^0 are the familiar ℓ_2 and L_2, respectively. We may also describe ℓ_2^m as the class of sequences obtained from elements of ℓ_2 by m successive summations. Similarly the elements of L_2^m are obtained from those of L_2 by m successive integrations.

Theorem 2. If

(16) $$(y_\nu) \in \ell_2^m,$$

then the interpolation problem

(17) $$S(\nu) = y_\nu \quad \text{for all } \nu,$$

has a unique solution such that

(18) $$S(x) \in \mathcal{S}_{2m-1} \cap L_2^m.$$

This solution $S(x)$ has the following extremum property: If $f(x)$ is an arbitrary function such that

(19) $$f(x) \in L_2^m$$

and

(20) $$f(\nu) = y_\nu \quad \text{for all } \nu,$$

then

(21) $$\int_{-\infty}^{\infty} |f^{(m)}(x)|^2 dx > \int_{-\infty}^{\infty} |S^{(m)}(x)|^2 dx,$$

unless $f(x) = S(x)$ for all real x.

In words: If $(y_\nu) \in \ell_2^m$, then the spline interpolant $S(x)$ minimizes the integral

(22) $$I(f) = \int_{-\infty}^{\infty} |f^{(m)}(x)|^2 dx$$

among all sufficiently smooth interpolants of (y_ν).

If $y_\nu = P(\nu)$ for all ν, where* $P(x) \in \pi_{m-1}$, then $P(x) \in \mathcal{S}_{2m-1} \cap L_2^m$, and therefore $S(x) = P(x)$ by the unicity of the solution in Theorem 2. However, here $I(S) = 0$. In the general case of $(y_\nu) \in \ell_2^m$ we may therefore say that $S(x)$ is among all interpolants of (y_ν), the one that "is most nearly" a polynomial of degree $\leq m - 1$.

* π_{m-1} is the class of polynomials of degree $m-1$ or less.

If $y_\nu = P(\nu)$, where $P(x) \in \pi_{2m-1}$, but $P(x) \notin \pi_{m-1}$, then again $P(x) \in \mathcal{S}_{2m-1} \cap PG$, and so $S(x) = P(x)$ is the unique solution of Theorem 1. Theorem 2 does not apply here because $(y_\nu) \notin \ell_1^m$. There is no interpolant $f(x)$ such that $I(f) < \infty$.

How do we actually construct the solutions $S(x)$ of these interpolation problems? To answer this question let us for the moment assume that

(23) $$(y_\nu) \in \ell_1, \text{ hence a fortiori } (y_\nu) \in \ell_2.$$

This insures the continuity of the periodic function

(24) $$T(u) = \sum_{-\infty}^{\infty} y_\nu e^{i\nu u}$$

which we call the generating function of the sequence (y_ν). Here and below we denote the relationship between a sequence and its generating function symbolically by writing

(25) $$(y_\nu) \mapsto T(u) .$$

We also require the generating function of the sequence $(M_{2m}(\nu))$, which is

(26) $$\Phi_{2m}(u) = \sum_{\nu=-(m-1)}^{m-1} M_{2m}(\nu) e^{i\nu u} .$$

This is a cosine polynomial of order $m-1$, because $M_{2m}(x) = 0$ if $|x| \geq m$. It is readily evaluated by (7), and we find that

$$\Phi_2(u) = 1,\ \Phi_4(u) = \frac{1}{3}(2+\cos u),\ \Phi_6(u) = \frac{1}{60}(33 + 26\cos u + \cos 2u),\ \cdots .$$

It also has the property that

(27) $$0 < \Phi_{2m}(\pi) \leq \Phi_{2m}(u) \leq \Phi_{2m}(0) = 1 \quad \text{for all } u.$$

It follows that its reciprocal has an expansion

(28) $$\frac{1}{\Phi_{2m}(u)} = \sum_{-\infty}^{\infty} \omega_\nu e^{i\nu u}$$

with real coefficients ω_ν, $\omega_{-\nu} = \omega_\nu$, that decay exponentially. Let us find the standard representation

(29) $$S(x) = \sum_j c_j M_{2m}(x-j)$$

of the solution of the interpolation problem (17), which requires that

(30) $$\sum_j c_j M_{2m}(\nu - j) = y_\nu \quad \text{for all } \nu.$$

Furthermore let

(31) $$C(u) = \sum_j c_j e^{iju}, \quad \text{or} \quad (c_j) \mapsto C(u),$$

be the as yet unknown generating function of the (c_j). Since the convolution of two sequences has a generating function that is the product of the generating functions of the two sequences, we see by (24), (26), and (31), that the relations (30) are equivalent to the relation

(32) $$C(u)\Phi_{2m}(u) = T(u), \quad \text{or} \quad C(u) = \frac{T(u)}{\Phi_{2m}(u)} .$$

Now (28) shows that $(c_\nu) = (y_\nu) * (\omega_\nu)$ and therefore

(33) $$c_\nu = \sum_j y_j \omega_{\nu-j} \quad \text{for all } \nu.$$

These are the coefficients of the interpolating spline (29).

Examples. 1. If $m = 1$, then $\Phi_2(u) = 1$, hence $\omega_0 = 1$, $\omega_\nu = 0$ $(\nu \neq 0)$, and we obtain $c_\nu = y_\nu$ for all ν. If $m = 2$, we find (See [9, Lecture 4, Section 5]) that

$$\omega_\nu = \sqrt{3} \cdot \lambda^{|\nu|}, \text{ where } \lambda = -2 + \sqrt{3} = -.26795.$$

2. If we choose $y_\nu = \delta_\nu$, where $\delta_0 = 1$, $\delta_\nu = 0(\nu \neq 0)$, then (33) shows that $c_\nu = \omega_\nu$. Therefore the spline

(34) $$L_{2m-1}(x) = \sum_{\nu} \alpha_{\nu} M_{2m}(x-\nu) .$$

is the solution of the interpolation problem

(35) $$L_{2m-1}(\nu) = \delta_{\nu}, \quad \text{for all } \nu.$$

The function (34) is the _fundamental function_ of the process, and _the solution_ $S(x)$ of _the general problem_ (17) _is given by_

(36) $$S(x) = \sum_{\nu=-\infty}^{\infty} y_{\nu} L_{2m-1}(x-\nu) .$$

This cardinal interpolation formula bridges the gap between the linear interpolant (3) and the cardinal series (2). In fact, notice that if $m = 1$ then (36) reduces to (3), while we have

(37) $$\lim_{m\to\infty} S_{2m-1}(x) = \frac{\sin \pi x}{\pi x} .$$

Also every derivative $S_{2m-1}^{(k)}(x)$ converges to the corresponding derivative of the right side of (37), uniformly for all real x.

In our discussion we have assumed that (23) holds. However, _the relations_ (33), (29), _and_ (36) _are valid for both Theorems_ 1 _and_ 2, _under their respective assumptions_.

Part II. _The cardinal smoothing spline_

1. _Statement of the problem_. We assume now that

(1) $$(y_{\nu}) \in \ell_1, \text{ hence } \Sigma|y_{\nu}| < \infty ,$$

and restrict ourselves to real-valued data and functions. We also recall the definitions (3.14) and (3.15) of Part I, of the classes ℓ_2^m and L_2^m. In view of the inclusion relations

(2) $$\ell_1 \subset \ell_2 \subset \ell_2^1 \subset \cdots \subset \ell_2^k \subset \ell_2^{k+1} \subset \cdots$$

(See [9, p. 104]), we observe that (1) implies that (y_{ν}) satisfies the assumptions of Theorem 2 for all m.

The problem. _We are given_ m _and a smoothing parameter_ $\epsilon > 0$. _Among all functions_

(3) $$f(x) \in L_2^m, \text{ hence } f^{(m)}(x) \in L_2 (\mathbb{R}),$$

we wish to find the solution of the problem

(4) $$J(f) \equiv \epsilon \cdot \int_{-\infty}^{\infty} (f^{(m)}(x))^2\, dx + \sum_{-\infty}^{\infty} (f(\nu) - y_{\nu})^2 = \text{minimum}.$$

Lemma 1. _In solving the minimum problem_ (4) we may restrict the choice _of admissible functions_ f(x) _to the elements of_

(5) $$\mathcal{S}_{2m-1} \cap L_2^m .$$

Proof: If f(x) is such that $J(f) < \infty$, then $(f(\nu) - y_{\nu}) \in \ell_2$. We apply Theorem 2 to the sequence $(f(\nu))$, and let

(6) $$s(x) \in \mathcal{S}_{2m-1} \cap L_2^m$$

be such that $s(\nu) = f(\nu)$ for all ν. But then

$$J(s) = \epsilon \int (s^{(m)})^2\,dx + \Sigma(s(\nu) - y_\nu)^2$$

$$= \epsilon \int (s^{(m)})^2\,dx + \Sigma(f(\nu) - y_\nu)^2 ,$$

and so

$$J(s) \leq \epsilon \int_{-\infty}^{\infty} (f^{(m)})^2\,dx + \Sigma(f(\nu) - y_\nu)^2 = J(f),$$

in view of the extremum property of $s(x)$ as expressed by (3.21) of Theorem 2. Therefore, for any $f(x)$, the spline $s(x)$ that interpolates $f(x)$, produces a value $J(s) \leq J(f)$.

Let us therefore find the solution

(7) $$S(x) = \sum_j c_j M_{2m}(x - j)$$

of the minimum problem

(8) $$J(S) = \epsilon \int_{-\infty}^{\infty} (S^{(m)})^2\,dx + \sum_{-\infty}^{\infty} (S(\nu) - y_\nu)^2 = \text{minimum}.$$

Here we need another

Lemma 2. If (7) satisfies $S(x) \in L_2(R)$, hence also $(c_j) \in \ell_2$, then

(9) $$\int_{-\infty}^{\infty} (S^{(m)}(x))^2\,dx = \sum_{j,\nu} \gamma_{j-\nu} c_j c_\nu ,$$

where

(10) $$(\gamma_\nu) \mapsto \left(2 \sin \frac{u}{2}\right)^2 \phi_{2m}(u).$$

Proof. From (7) we find that

(11) $$\int_{-\infty}^{\infty} (S^{(m)}(x))^2\,dx = \int \Big\{ \sum_{j,k} c_j c_k M_{2m}^{(m)}(x - j)\, M_{2m}^{(m)}(x - k) \Big\}\,dx$$

$$= \sum_{j,k} \gamma_{j-k} c_j c_k ,$$

where (γ_r) is the even sequence defined by

(12) $$\gamma_r = \int_{-\infty}^{\infty} M^{(m)}(x)\, M^{(m)}(x - r)\,dx ,$$

where, to simplify notations we dropped the subscript $2m$ of $M_{2m}(x)$. Integrations by parts show that

(13) $$\gamma_r = (-1)^{m-1} \int_{-\infty}^{\infty} M'(x)\, M^{(2m-1)}(x - r)\,dx .$$

Observe that $M^{(2m-1)}(x)$ is a step function assuming in consecutive unit intervals the values

(14) $$\cdots, 0, 0, 1, -\binom{2m-1}{1}, \binom{2m-1}{2}, \cdots, -1, 0, 0, \cdots .$$

This sequence has the generating function

(15) $$\sum_{j=0}^{2m-1} (-1)^j \binom{2m-1}{j} e^{iju} = (1 - e^{iu})^{2m-1},$$

except for a shift factor e^{iuk} which we disregard. Now (13) indicates that (γ_r) is the convolution of the sequence (14) with the sequence

$$\int_\nu^{\nu+1} M'(x)\,dx = M(\nu+1) - M(\nu) \mapsto (e^{-iu} - 1)\,\phi_{2m}(u).$$

However, in (13) the sequence (γ_r) appears as a sum of the form $\sum_\nu a_\nu b_{\nu-r}$. If we pass from (a_ν) to the reversed sequence $(a_{-\nu})$, we obtain a genuine convolution $\sum_\nu a_{-\nu} b_{\nu-r}$. Let us therefore reverse the first sequence (14). As we obtain the generating function of the reversed sequence by changing u into $-u$ in its original generating function, we find the generating function of (γ_r) to be (up to a shift factor e^{iuk}) the product

$$(-1)^{m-1}(1-e^{-iu})^{2m-1}\cdot(e^{-iu}-1)\,\phi_{2m}(u)$$

$$=(-1)^m(1-e^{-iu})^{2m}\,\phi_{2m}(u)$$

$$=e^{-ium}(-1)^m(e^{iu/2}-e^{-iu/2})^{2m}\,\phi_{2m}(u)$$

$$=e^{-ium}(2\sin\tfrac{u}{2})^{2m}\phi_{2m}(u).$$

Since (γ_r) is an <u>even</u> sequence, its generating function must be even, and therefore

$$(\gamma_r)\mapsto(2\sin\tfrac{u}{2})^{2m}\phi_{2m}(u),$$

establishing (10).

2. <u>Solution of the problem</u>. From (8), (9), and (7) we find that

(1) $$J(S)=\epsilon\sum_{j,\nu}\gamma_{j-\nu}c_jc_\nu+\sum_\nu\{\sum_j c_jM(\nu-j)-y_\nu\}^2.$$

Let us minimize this function of the (c_k). To obtain the normal equations, we differentiate $J(S)$ obtaining

$$\frac{1}{2}\frac{\partial}{\partial c_k}J(S)=\epsilon\sum_j\gamma_{j-k}c_j+\sum_\nu\{\sum_j c_jM(\nu-j)-y_\nu\}M(\nu-k)=0\quad(k\in Z).$$

If we sum within the double-sum only with respect to ν, we obtain

(2) $$\sum_\nu M_{2m}(\nu-j)M_{2m}(\nu-k)=\alpha_{j-k}$$

where

(3) $$(\alpha_\nu)\mapsto(\phi_{2m}(u))^2.$$

The normal equations thus become

$$\epsilon\sum_j\gamma_{j-k}c_j+\sum_j\alpha_{j-k}c_j=\sum y_\nu M_{2m}(\nu-k)\quad(k\in Z),$$

or

(4) $$\sum_j(\alpha_{j-k}+\epsilon\gamma_{j-k})c_j=\sum_\nu y_\nu M_{2m}(\nu-k)\quad(k\in Z).$$

However, by (3) and (1.10) we find

(5) $$(\alpha_\nu+\epsilon\gamma_\nu)\mapsto(\phi_{2m}(u))^2+\epsilon(2\sin\tfrac{u}{2})^{2m}\phi_{2m}(u).$$

From

$$(M_{2m}(\nu))\mapsto\phi_{2m}(u),$$

and writing

(6) $$(c_\nu)\mapsto C(u),\ (y_\nu)\mapsto T(u),$$

we find the normal equations (4) to be equivalent to the relation

$$\left\{(\phi_{2m}(u))^2+\epsilon(2\sin\tfrac{u}{2})^{2m}\phi_{2m}(u)\right\}C(u)=T(u)\,\phi_{2m}(u),$$

whence

(7) $$(c_\nu)\mapsto C(u)=\frac{T(u)}{\phi_{2m}(u)+\epsilon(2\sin\frac{u}{2})^{2m}}.$$

This establishes

<u>Theorem 3</u>. <u>In terms of the expansion</u>

(8) $$\frac{1}{\phi_{2m}(u)+\epsilon(2\sin\frac{u}{2})^{2m}}=\sum_{\nu=-\infty}^{\infty}\omega_\nu(\epsilon)e^{i\nu u},$$

<u>where the coefficients</u> $\omega_\nu(\epsilon)=\omega_{-\nu}(\epsilon)$ <u>decay exponentially</u>, <u>the coefficients</u> (c_j) <u>of the solution</u>

(9) $$S(x)=\sum_j c_jM_{2m}(x-j)$$

of the minimum problem, are

$$c_\nu = \sum_{j=-\infty}^{\infty} y_j \omega_{\nu-j}(\epsilon). \tag{10}$$

We call the solution (9) the cardinal smoothing spline.

3. A few properties of the cardinal smoothing spline $S(x) = S(x;\epsilon)$.

A. We have assumed above that $\epsilon > 0$. However, if we set $\epsilon = 0$ in (2.8), it becomes

$$\frac{1}{\phi_{2m}(u)} = \sum_\nu \omega_\nu(0) e^{i\nu u}, \tag{1}$$

and a comparison with the expansion (3.28) of Part I, shows that $\omega_\nu(0) = \omega_\nu$ for all ν: This shows that $S(x;0) = S(x)$ reduces to the interpolating cardinal spline $S(x)$ of Theorem 2.

B. What is the effect of the smoothing spline $S(x) = S(x;\epsilon)$ on the original data (y_ν)? This we answer by determining the "smoothed" sequence $(S(\nu))$, to compare it with (y_ν). By (2.9) and (2.10) we find

$$S(\nu)) = (c_\nu) * (M_{2m}(\nu)),$$

and therefore, by (2.7),

$$(S(\nu)) \longmapsto C(u)\, \phi_{2m}(u) = \frac{T(u)}{1 + \epsilon \dfrac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}}\, . \tag{2}$$

In terms of the expansion

$$\frac{1}{1 + \epsilon \dfrac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}} = \sum_\nu \sigma_\nu(\epsilon) e^{i\nu u} \tag{3}$$

(2) shows that the sequence $(S(\nu;\epsilon))$ arises from the data (y_ν) by the smoothing formula

$$S(\nu;\epsilon) = \sum_j \sigma_{\nu-j}(\epsilon) y_j. \tag{4}$$

Observe that by (2.8) and (3) the coefficients $\sigma_\nu(\epsilon)$ are expressed in terms of $\omega_\nu(\epsilon)$ by $\sigma_\nu(\epsilon) = \sum_j M_{2m}(\nu - j)\, \omega_j(\epsilon)$.

Is (4) a smoothing formula according to our definition of Part I, Section 1? That it is one we see if we inspect its characteristic function

$$K(u;\epsilon) = \frac{1}{1 + \epsilon \dfrac{(2 \sin \frac{u}{2})^{2m}}{\phi_{2m}(u)}}\, , \tag{5}$$

for it is evident that

$$0 < K(u;\epsilon) < K(0;\epsilon) = 1 \quad \text{for} \quad 0 < u < 2\pi\, . \tag{6}$$

C. The smoothing power of the formula (4) increases with increasing ϵ. In [4, Definition 2, p. 53] we gave good reasons for the following definition: Of two different smoothing formulae having the characteristic functions $\phi(u)$ and $\tilde{\phi}(u)$, we say that the second has greater smoothing power, provided that

$$|\tilde{\phi}(u)| \leq |\phi(u)| \quad \text{for all } u, \text{ excluding equality for all } u. \tag{7}$$

However, if $0 < \epsilon < \tilde{\epsilon}$, it is clear by (5) that

$$0 < K(u;\tilde{\epsilon}) < K(u;\epsilon) \quad \text{if} \quad 0 < u < 2\pi,$$

and the criterion (7) is satisfied.

D. The degree of exactness of the smoothing formula (4) is $= 2m - 1$. This follows from (1.7) of Part I, because (5) shows that we have the expansion in powers of u

$$K(u;\epsilon) = 1 - \epsilon u^{2m} + \cdots\, . \tag{8}$$

E. If we drop our assumption (1.1), and assume only that (y_ν) is of power growth, then our construction of the smoothing spline $S(x) = S(x;\epsilon)$ by the formulae (2.8), (2.10), and (2.9), remains applicable. Of course, its earlier connection with the functional $J(S)$, of (1.8), no longer holds. In fact we will find that $J(S) = \infty$ for all splines S. Presumably, it is still true that our $S(x;\epsilon)$ minimizes $J(S)$, provided that (y_ν) satisfies the condition

$$(y_\nu) \in \ell_2^m$$

of Theorem 2. However, this I was not able to establish.

In any case I recommend the cardinal smoothing spline $S(x;\epsilon)$, which represents the modification, found more than 30 years later, of my war-time approach to the problem of cardinal smoothing.

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Department of Mathematics
United States Military Academy
West Point, New York 10996

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The paper has two parts. In part I it describes the old results on the problem of smoothing a given bi-infinite sequence of equidistant data ~~([4], [5], [1], [2])~~. It includes ~~in~~ §3 the main results on the cardinal spline interpolation of equidistant data of power growth. All this is meant as background for the new developments in Part II.

In Part II we present a method of smoothing a sequence of equidistant data, ~~(y_ν) $(-\infty < \nu < \infty)$~~. It is based on the ideas of E.T. Whittaker and may be

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Abstract continued.

described as follows. In terms of the central B-spline $M_{2m}(x)$, defined by (7) of Part I, §3, we define the cosine polynomial $\phi_{2m}(u)$ by (26) of Part I, §3. Given a positive smoothing parameter ε, we define the sequence of "weights" $\{\omega_\nu(\varepsilon)\}$ by the Fourier expansion

$$\frac{1}{\phi_{2m}(u) + \varepsilon(2 \sin \frac{u}{2})^{2m}} = \sum_{-\infty}^{\infty} \omega_\nu(\varepsilon) e^{i\nu u} .$$

Let $\sum_{-\infty}^{\infty} |y_\nu| < \infty$. It is then shown that the cardinal spline

$$S(x;\varepsilon) = \sum_{-\infty}^{\infty} c_j M_{2m}(x-j),$$

where

$$c_j = \sum_{\nu=-\infty}^{\infty} y_\nu \omega_{j-\nu}(\varepsilon) ,$$

is the solution of the following minimum problem: Among all functions $f(x)$ such that

$$f^{(m)}(\varepsilon) \in L_2(\mathbb{R})$$

to find $f(x)$ such that the functional

$$J(f) \equiv \varepsilon \int_{-\infty}^{\infty} (f^{(m)}(x))^2 dx + \sum_{-\infty}^{\infty} (f(\nu) - y_\nu)^2$$ is to be minimal.